Math 246B Lecture 25 Notes

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1 Fatou's Theorem and the Riesz-Herglotz Theorem

1.1 Fatou's theorem, continued

Last time, we were in the middle of proving Fatou's theorem.

Theorem 1.1 (Fatou). *Let u be harmonic in D and bounded. Then the radial limits $\lim_{r \to 1^-} u(rz)$ exist for a.e. $z \in \partial D$ (with respect to 1-dimensional) Lebesgue measure on the circle. If $u = f \in \mathrm{Hol}(D)$ and $f(z) = \lim_{r \to 1^-} f(rz)$ vanishes on a set of positive measure (on the circle), then $f \equiv 0$.*

Proof. We have shown that there exists $g \in L^\infty(\partial D)$ such that

$$u = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(z, e^{i\varphi}) g(e^{i\varphi}) \, d\varphi.$$

Let $e^{i\alpha} \in \partial D$ be a Lebesgue point of g:

$$\frac{1}{2\pi\rho} \int_{\alpha-\rho}^{\alpha+\rho} |g(e^{i\varphi}) - g(e^{i\alpha})| \, d\varphi \to 0.$$

We claim that the radial limit $\lim_{r \to 1} u(re^{i\alpha})$ exists an equals $g(e^{i\alpha})$. This will establish the theorem, as a.e. point in ∂D is a Lebesgue point of g. We can assume that $\alpha = 0$ and that $g(e^{i\alpha}) = 0$ (otherwise consider $u(e^{i\alpha z}) - g(e^{i\alpha})$). Thus,

$$\frac{1}{2\pi\rho} \int_{-\rho}^{\rho} |g(e^{i\varphi})| \, d\varphi \to 0,$$

and we want to show that $u(x) \to 0$ as $x \to 1^-$ along $\mathbb{R}$.

Plugging in the formula for the Poisson kernel,

$$u(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - x^2}{|x - e^{i\varphi}|^2} g(e^{i\varphi}) \, d\varphi.$$

The contribution to this integral coming from $\int_{\pi/2\le|\varphi|\le\pi} \to 0$, as $P(x,e^{i\varphi}) \to 0$ uniformly in φ. Estimate the contribution from $|\varphi| \le \pi/2$: Writing $\delta = 1-x$,

$$P(x,e^{i\varphi}) = \frac{1-(1-\delta)^2}{|x-e^{i\varphi}|^2} = \frac{2\delta-\delta^2}{(x-\cos(\varphi))^2+\sin^2(\varphi)} \le \frac{2\delta}{\sin^2(\varphi)} \le \frac{2\delta}{\varphi^2}.$$

We get

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-x^2}{|x-e^{i\varphi}|^2}|g(e^{i\varphi})|\,d\varphi &\le \int_{A\delta\le|\varphi|\le\pi/2} + \int_{|\varphi|\le A\delta} \\
&\le \int_{A\delta\le|\varphi|\le\pi/2} \frac{2\delta}{\varphi^2}|g(e^{i\varphi})|\,d\varphi + \int_{|\varphi|\le A\delta} \frac{2\delta-\delta^2}{|x-e^{i\varphi}|^2}|g(e^{i\varphi})|\,d\varphi \\
&\le \int_{A\delta\le|\varphi|\le\pi/2} \frac{2\delta}{\varphi^2}|g(e^{i\varphi})|\,d\varphi + \int_{|\varphi|\le A\delta} \frac{2\delta}{\delta^2}|g(e^{i\varphi})|\,d\varphi \\
&\le \frac{C\delta}{A\delta} + \frac{2}{\delta}\int_{|\varphi|\le A\delta} |g(e^{i\varphi})|\,d\varphi.
\end{aligned}$$

Given $\varepsilon > 0$, take A large so that $C/A \le \varepsilon$ for all $0 < \delta \le \delta_0(\varepsilon)$. For δ small enough, $\int_{|\varphi|\le\pi/2} P(x,e^{i\varphi})|g(e^{i\varphi})|\,d\varphi \le 7\varepsilon$. Thus, $u(x) \to 0$ as $x \to 1^-$. Thus, for a.e. $z \in \partial D$, $\lim_{r\to1} u(rz)$ exists and equals $g(z)$.

For the latter part of the theorem, assume now that $f \in \mathrm{Hol}(D)$ is bounded. Then for a.e. $z \in \partial D$, $\lim_{r\to1} f(rz) =: f(z) \in L^\infty(\partial D)$. We claim that if $f(z) = 0$ on a set of positive measure in ∂D, then $f(z) \equiv 0$ in $|z| < 1$. The function $\log|f|$ is subharmonic in D, so

$$r \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\varphi})|\,d\varphi$$

is an increasing function. For any $0 < r < 1$, using Fatou's lemma,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\varphi})|\,d\varphi &\le \limsup_{r\to1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\varphi})|\,d\varphi \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{i\varphi})|\,d\varphi.
\end{aligned}$$

If $f \not\equiv 0$, we can conclude that the integral $> -\infty$. So $\log|f| \in L^1(\partial D)$, so $\{f = 0\}$ is a Lebesgue null set in ∂D. □

1.2 Representing harmonic functions by measures

We have been looking at functions u such that

$$u(z) = \frac{1}{2\pi}\int_{|w|=1} P(z,w)g(q)\,|dw|$$

for some $g \in L^\infty$. Let's try to replace $g \in L^\infty$ by $g \in L^1$ or by a (Borel, regular, Radon) measure $d\mu$ on ∂D.

Theorem 1.2 (F. Riesz-Herglotz)**.** *Let μ be a measure on ∂D, and let*

$$u = \int_{|w|=1} P(z,w)\, d\mu(w), \qquad |z| < 1.$$

Then u is harmonic in D, and the function $r \mapsto \int_{|z|=1} |u(rz)|\,|dz|$ is bounded on $[0,1)$. If $u_r(z) = u(rz)$, then $u_r \xrightarrow{r \to 1} \mu$ in the following weak sense: for any $\varphi \in C(\partial D)$,

$$\frac{1}{2\pi} \int_{|z|=1} u_r(z)\varphi(z)\,|dz| \xrightarrow{r\ to 1} \int_{|z|=1} \varphi(z)\, d\mu(z).$$

Conversely, let u be harmonic in D such that $\int_{|z|=1} |u(rz)|\,|dz| \leq C$ for all $0 \leq r \leq 1$. Then there exists a unique measure μ on ∂D such that

$$u(z) = \int_{|w|=1} P(z,w)\, d\mu(w) \qquad |z| < 1.$$

Moreover, $u_r \to \mu$ in the same weak sense.

Example 1.1. Let $u \geq 0$ be harmonic. Then the theorem applies, so

$$u(z) = \int_{|w|=1} P(z,w)\, d\mu(w),$$

where μ is a positive measure.